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APPROXIMATE POINT SPECTRUM OF A WEIGHTED SHIFT

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Notation. If T is a Hilbert space operator, let $\Lambda(T)$ denote its spectrum, $\Pi(T)$ its approximate point spectrum, $\Pi_0(T)$ its point spectrum, $\Gamma(T)$ its compression spectrum, $m(T)$ its lower bound (i.e., $\inf\{\|Tx\|/\|x\| : x\neq 0\}$), and $r(T)$ its spectral radius. Let $i(T)$ denote $\sup_n m(T^n)^{1/n}$, which equals $\lim_n m(T^n)^{1/n}$.

Let R denote a weighted right shift on l^2_+, defined by $Re_n=s_ne_{n+1}$, where (e_n) is an orthonormal basis of l^2_+, $n=1, 2, \ldots$. Let L denote its adjoint, a weighted left shift. Let B denote a weighted two-sided shift on l^2, defined by $Be_n=s_ne_{n+1}$, $n=0, \pm 1, \pm 2, \ldots$, (e_n) here being an orthonormal basis of l^2. If B has purely nonzero weights (s_n), then let

$$i(B)^- = \lim_n \inf_{k\leqq 0} |s_{k-1}\cdots s_{k-n}|^{1/n}, \qquad i(B)^+ = \lim_n \inf_{k\geqq -1} |s_{k+1}\cdots s_{k+n}|^{1/n},$$

$$r(B)^- = \lim_n \sup_{k\leqq 0} |s_{k-1}\cdots s_{k-n}|^{1/n}, \qquad r(B)^+ = \lim_n \sup_{k\geqq -1} |s_{k+1}\cdots s_{k+n}|^{1/n}.$$

Background. Λ and its parts, for weighted shifts, are known to have circular symmetry about 0; $\Pi_0(R)$ is known to be empty or $\{0\}$; $\Gamma(R)$ is known to be a disk, possibly degenerating to $\{0\}$; and $\Gamma(B)$ and $\Pi_0(B)$ are known to be annuli, possibly degenerate or empty, and in any case disjoint. These facts are easy to verify, and are proved in [3]. Also proved there is the much deeper fact that the spectrum of a weighted shift is always connected. This will also be deduced in this paper as an easy corollary of the results on the approximate point spectrum. Some of these results, from a different approach, seem to have been concurrently proved in [1].

The following formulae are easy to verify [3]:

$$i(R) = \lim_n \inf_k |s_{k+1}\cdots s_{k+n}|^{1/n}, \qquad r(R) = \lim_n \sup_k |s_{k+1}\cdots s_{k+n}|^{1/n}.$$

Preliminaries. The following propositions can be verified by routine arguments [4].

(1) Suppose (for R) that no s_n vanishes and ε, M are positive numbers. Then there are integers k and n, both greater than M, such that $|s_{k+1}\cdots s_{k+n}|^{1/n} > r(R)-\varepsilon$.

(2) For any positive numbers A, B, C, D,

$$\frac{A+B}{C+D} \leqq \text{maximum}\left(\frac{A}{C}, \frac{B}{D}\right).$$

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(3) Suppose (p_n) is a real nonnegative periodic sequence having period r, and $q=(p_1\cdots p_r)^{1/r}$. Suppose (a_n) is a real nonnegative sequence such that $\lim_n (a_n-p_n)=0$. If either some p_n vanishes, or no a_n vanishes, then

$$\lim_n \inf_k (a_{k+1}\cdots a_{k+n})^{1/n} = \lim_n \sup_k (a_{k+1}\cdots a_{k+n})^{1/n} = q.$$

(4) Suppose (for B) that no s_n vanishes. Then $i(B)=\text{minimum}\ (i(B)^-, i(B)^+)$ and $r(B)=\text{maximum}\ (r(B)^-, r(B)^+)$.

(5) For any operator T, $\Pi(T)\subset\{c : i(T)\leqq |c|\leqq r(T)\}$.

THEOREM 1. *If no s_n vanishes, then $\Pi(R)=\{c : i(R)\leqq |c|\leqq r(R)\}$. If finitely many s_n vanish, then $\Pi(R)=\{0\}\cup\Pi(R')$, where R' is the right shift with weights s_{k+1}, $s_{k+2},\ldots$, where s_k is the last zero weight. If infinitely many s_n vanish, then*

$$\Pi(R) = \{c : |c| \leqq r(R)\}.$$

Proof. If $i(R)=r(R)$, then $\Pi(R)$ is by (5) contained in, hence by nonemptiness and circular symmetry equal to, $\{c : |c|=r(R)\}$.

Suppose no s_n vanishes, and $i(R)<c<r(R)$. Since $\Pi(R)$ is closed and has circular symmetry, it suffices in view of (5), for the first assertion of the theorem, to show that c is necessarily in $\Pi(R)$.

Choose numbers a, b such that $i(R)<a<c<b<r(R)$, and suppose $\varepsilon>0$. Choose n such that $(c/b)^n<\varepsilon$ and k such that $|s_{k+1}\cdots s_{k+n}|^{1/n}>b$. By (1) choose p such that $(a/c)^p<\varepsilon$ and m such that $m>n+k$ and $|s_{m+1}\cdots s_{m+p}|^{1/p}<a$.

Define $x=(x_i)$ by

$$x_{k+1} = 1,$$

$$x_r = \frac{s_{k+1}\cdots s_{r-1}}{c^{r-k-1}} \quad \text{if } k+2 \leqq r \leqq m+p+1,$$

$$x_r = 0 \quad \text{if } r < k+1 \text{ or } r > m+p+1.$$

Then

$$Rx-cx = \sum_{r=k+1}^{m+n+1}\left(\frac{s_{k+1}\cdots s_r}{c^{r-k-1}}\, e_{r+1}-\frac{s_{k+1}\cdots s_{r-1}}{c^{r-k}}\, e_r\right)$$

$$= s_{m+p+1}x_{m+p+1}e_{m+p+2}-ce_{k+1},$$

and hence

$$\|Rx-cx\|^2 = |s_{m+p+1}|^2|x_{m+p+1}|^2+c^2$$
$$\leqq \|R\|^2(1+|x_{m+p+1}|^2).$$

Also

$$\|x\|^2 = \sum |x_i|^2 \geqq |x_{k+n+1}|^2+|x_{m+1}|^2.$$

But

$$|x_{k+n+1}| = |s_{k+1}\cdots s_{k+n}|/c^n > (b/c)^n > 1/\varepsilon$$

and

$$|x_{m+p+1}/x_{m+1}| = |s_{m+1}\cdots s_{m+p}|/c^n < (a/c)^p < \varepsilon.$$

So by (2),

$$\begin{aligned}\frac{\|Rx-cx\|^2}{\|x\|^2} &\leqq \|R\|^2 \frac{1+|x_{m+p+1}|^2}{|x_{k+n+1}|^2+|x_{m+1}|^2} \\ &\leqq \|R\|^2 \max\left(\frac{1}{|x_{k+n+1}|^2}, \left|\frac{x_{m+p+1}}{x_{m+1}}\right|^2\right) \\ &< \varepsilon^2\|R\|^2\end{aligned}$$

and so c is in $\Pi(R)$.

Now suppose infinitely many s_n vanish, and suppose $0<c<r(R)$. The last assertion of the theorem will follow if we show that, necessarily, c is in $\Pi(R)$.

Choose b such that $c<b<r(R)$. Given $\varepsilon>0$, choose n such that $(c/b)^n<\varepsilon$ and k such that $|s_{k+1}\cdots s_{k+n}|^{1/n}>b$. Let r be the first index greater than $k+n$ such that $s_r=0$. Define $x=(x_i)$ by

$$\begin{aligned} x_{k+1} &= 1, \\ x_m &= \frac{s_{k+1}\cdots s_{m-1}}{c^{m-k-1}} \quad \text{if } k+2 \leqq m \leqq r, \\ x_m &= 0 \quad \text{if } m \leqq k \text{ or } m > r. \end{aligned}$$

Then $Rx-cx=ce_{k+1}$, $\|Rx-cx\|=c$, and

$$\|x\| \geqq |x_{k+n+1}| = |s_{k+1}\cdots s_{k+n}|/c^n > (b/c)^n > 1/\varepsilon$$

so $\|Rx-cx\|/\|x\|<c\varepsilon$, and c is in $\Pi(R)$.

If finitely many s_n vanish, then R is the orthogonal sum of R' and a nilpotent operator, and $\Pi(R)=\Pi(R')\cup\{0\}$; applying the earlier argument for nonzero weights to R', we have the second assertion.

Corollary (Kelley). $\Lambda(R)=\{c : |c|\leqq r(R)\}$.

Proof. Π contains the boundary of Λ, which is therefore either the annulus Π or the closed disk of radius $r(R)$; it is the latter since 0 is in $\Gamma(R)$.

Definition. A sequence (a_n) is *almost periodic* if there is a periodic sequence (p_n) such that $\lim_n (a_n-p_n)=0$. If r is the period of (p_n), the *periodic mean* is $(p_1\cdots p_r)^{1/r}$.

Theorem 2. *If $(|s_n|)$ is almost periodic, then $\Pi(R)=\{c : |c|=q\}$ if all s_n are nonzero, and is the same set together with $\{0\}$ if some s_n vanishes; in either case $\Lambda(R)=\{c : |c|\leqq q\}$, where q is the periodic mean of the approximating periodic sequence (p_n).*

Proof. The last statement follows from the first two by the corollary to Theorem 1.

If either some p_n vanishes or no s_n vanishes, then $i(R)=r(R)=q$ by (3), and $\Pi(R)$ is as asserted by Theorem 1.

Suppose no p_n vanishes, but some s_n vanishes. Then only finitely many s_n vanish, since $\lim_n (|s_n|-p_n)=0$ and (p_n), assuming only finitely many distinct values, is bounded away from 0. Theorem 1 now applies again, and $\Pi(R)$ is as asserted.

COROLLARY 1. *If R is injective and $|s_n| \to s$, then*

$$\Pi(R) = \{c : |c| = s\} \quad \text{and} \quad \Lambda(R) = \{c : |c| \leqq s\}.$$

COROLLARY 2. *If $(|s_n|)$ is periodic with mean q, then*

$$\Pi(R) = \{c : |c| = q\} \quad \text{and} \quad \Lambda(R) = \{c : |c| \leqq q\}.$$

EXAMPLE. Let $T=$subdiagonal $(1, 2, 1, 2, \ldots)$. By Corollary 2,

$$\Pi(T) = \{c : |c| = \sqrt{2}\} \quad \text{and} \quad \Lambda(T) = \{c : |c| \leqq \sqrt{2}\}.$$

NOTE. If some $s_n=0$ for B, then B is the orthogonal sum of a right and a left shift, and their approximate point spectra are described elsewhere in this paper. In treating B below, we therefore assume that no s_n vanishes.

THEOREM 3. *If $r(B)^- < i(B)^+$, then*

$$\Pi(B) = \{c : i(B)^+ \leqq |c| \leqq r(B)^+ \text{ or } i(B)^- \leqq |c| \leqq r(B)^-\}.$$

Otherwise $\Pi(B)=\{c : i(B) \leqq |c| \leqq r(B)\}$.

Proof. Since $\Pi(B)$ is closed and has circular symmetry, in view of (4) and (5) we need only consider positive c lying between any two of the values $i(B)^-$, $i(B)^+$, $r(B)^-$ and $r(B)^+$.

If $i(B)^+ < c < r(B)^+$, then exact imitation of the construction of Theorem 1 (with $i(B)^+ < a < c < b < r(B)^+$) yields approximate eigenvectors for c. So

$$\{c : i(B)^+ \leqq |c| \leqq r(B)^+\} \subset \Pi(B).$$

Suppose $i(B)^- < c < r(B)^-$. Choose numbers a, b such that $i(B)^- < a < c < b < r(B)^-$. Choose p such that $(a/c)^p < \varepsilon$ and $m < -p$ such that $|s_{m+1} \cdots s_{m+p}|^{1/p} < a$. Choose n such that $(c/b)^n < \varepsilon$ and, by (1), choose $k < m-n$ such that $|s_{k+1} \cdots s_{k+n}|^{1/n} > b$. Define x as in the proof of Theorem 1, and again we find $c \in \Pi(B)$. So

$$\{c : i(B)^- \leqq |c| \leqq r(B)^-\} \subset \Pi(B).$$

Suppose $r(B)^+ < c < i(B)^-$. Choose a, b, n, p as before; choose $k < -n$ such that $|s_{k+1} \cdots s_{k+n}|^{1/n} > b$, and $m \geqq 0$ such that $|s_{m+1} \cdots s_{m+p}|^{1/p} < a$. Proceeding as in the proof of Theorem 1, we find that c is in $\Pi(B)$. So if $r(B)^+ \leqq i(B)^-$, then

$$\{c : r(B)^+ \leqq |c| \leqq i(B)^-\}$$

is contained in $\Pi(B)$.

Suppose $r(B)^- < c < i(B)^+$. We show that c is not in $\Pi(B)$. Suppose it were. Choose b strictly between c and $i(B)^+$. Then for some N we have, for all $n \geqq N$ and $k \geqq 1$, $|s_{k+1} \cdots s_{k+n}|^{1/n} > b$. For all positive ε, choose a unit vector $x = x(\varepsilon)$ such that $\|Bx - cx\| < \varepsilon$.

Suppose there exists a sequence $\varepsilon' \to 0$ such that $x_0(\varepsilon')$ (the 0th coefficient of $x(\varepsilon')$) converges to 0. Then for any positive ε we have, for some choice of ε', $\|Bx - cx\| < \varepsilon$ and $|x_0| < \varepsilon$. Let $x^0 = x_0 e_0$ and $x^1 = x - x^0$. Then

$$\|Bx^1 - cx^1\| \leqq \|Bx - cx\| + \|Bx^0 - cx^0\| \leqq \varepsilon + 2\|B\|\varepsilon.$$

We may therefore choose approximating eigenvectors $x(\varepsilon)$ such that x_0 always vanishes.

For such x, let

$$x^- = \sum_{n<0} x_n e_n, \qquad x^+ = x - x^- = \sum_{n>0} x_n e_n.$$

Then Bx^- and cx^- are both orthogonal to Bx^+ and cx^+, so $Bx-cx$ is the orthogonal sum of $Bx^- - cx^-$ and $Bx^+ - cx^+$. Both of the latter have norms therefore less than ε. Either x^- or x^+ has norm at least 1/2. It follows that approximate eigenvectors can be chosen from either 1^2_+ or 1^2_-. In the former case c is in $\Pi(R^+)$ where R^+ is the right shift having the positively indexed weights of B. By direct comparison of formulae (in terms of s_n), $i(B)^+ = i(R^+) \leqq |c| \leqq r(R^+) = r(B)^+$, contrary to hypothesis.

In the latter case, c is in $\Pi(L^-)$, where L^- is the left shift with weights $t_n = s_{-n}$. Then $|c| \leqq r(L^-) = r(B)^-$, again a contradiction.

So there is a positive number d such that, for some sequence $\varepsilon' \to 0$, $|x_0(\varepsilon')| \geqq d$ for all ε'.

If $n \geqq N$ then

$$|s_0 \cdots s_{n-1} x_0|/c^n > (b/c)^n d.$$

Also, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
\left| x_n - \frac{s_0 \cdots s_{n-1} x_0}{c^n} \right| &\leqq \left| x_n - \frac{s_{n-1} x_{n-1}}{c} \right| + \left| \frac{s_{n-1} x_{n-1}}{c} - \frac{s_{n-1} s_{n-2} x_{n-2}}{c^2} \right| \\
&\quad + \cdots + \left| \frac{s_{n-1} \cdots s_1 x_1}{c^{n-1}} - \frac{s_{n-1} \cdots s_0 x_0}{c^n} \right| \\
&\leqq \frac{1}{c} (|cx_n - s_{n-1} x_{n-1}|^2 + \cdots + |cx_1 - s_0 x_0|^2)^{1/2} \\
&\quad \cdot \left(1 + \left| \frac{s_{n-1}}{c} \right|^2 + \cdots + \left| \frac{s_{n-1} \cdots s_1}{c^{n-1}} \right|^2 \right)^{1/2} \\
&\leqq \frac{1}{c} \|Bx - cx\| \left(1 + \left[\frac{\|B\|}{c} \right]^2 + \cdots + \left[\frac{\|B\|}{c} \right]^{2(n-1)} \right)^{1/2} \\
&< \frac{\varepsilon}{c} \left[\frac{(\|B\|/c)^{2n} - 1}{(\|B\|/c)^2 - 1} \right]^{1/2} . \quad (*)
\end{aligned}$$

Fix $n \geqq N$ such that $(b/c)^n d > 2$, then choose $\varepsilon > 0$ such that $(*) \leqq 1$. We then have

$$|x_n| \geqq \left| \frac{s_0 \cdots s_{n-1} x_0}{c^n} \right| - \left| \frac{s_0 \cdots s_{n-1} x_0}{c^n} - x_n \right| > 1$$

which is impossible since x is a unit vector.

Therefore c is not in $\Pi(B)$, and the theorem now follows.

COROLLARY. *Either* $\Pi(B)$ *or* $\Pi(B^*)$ *is connected.*

Proof. Suppose $\Pi(B)$ is disconnected. Then $i(B^*)^+ \leqq r(B^*)^+ = r(B)^- < i(B)^+ = i(B^*)^- \leqq r(B^*)^-$ so $\Pi(B^*)$ is connected.

THEOREM 4. *If $\Pi_0(B)$ is nonempty, then $\Pi(B)$ is connected. If $\Pi(B)$ is disconnected, then $\Gamma(B)$ is an annulus whose boundary components are contained in distinct components of $\Pi(B)$.*

Proof. If $\Pi_0(B)$ is nonempty, then by [3] (or straightforward computation), it is an annulus, centered at 0, of inner radius $p_1(B) = \lim\sup_n |s_1 \cdots s_n|^{1/n}$ and outer radius $p_2(B) = \lim\inf_n |s_{-1} \cdots s_{-n}|^{1/n}$; by direct substitution in terms of s_i, and standard inequalities among various limits, we have $i(B)^+ \leqq p_1(B) \leqq p_2(B) \leqq r(B)^-$; $\Pi(B)$ is then connected by Theorem 3.

Also, $\Gamma(B)$ is an annulus of inner radius $c_1(B) = \lim\sup_n |s_{-1} \cdots s_{-n}|^{1/n}$ and outer radius $c_2(B) = \lim\inf_n |s_1 \cdots s_n|^{1/n}$ provided that $c_1(B) \leqq c_2(B)$. If $\Pi(B)$ is disconnected, then by Theorem 3 and standard inequalities among limits, $i(B)^- \leqq c_1(B) \leqq r(B)^- < i(B)^+ \leqq c_2(B) \leqq r(B)^+$. The second assertion now follows.

COROLLARY (KELLEY). $\Lambda(B)$ is connected.

Proof. $\Lambda = \Pi \cup \Gamma$.

THEOREM 5. *Suppose $(|s_n|)$ and $(|s_{-n}|)$, $n > 0$, are almost periodic with approximating periodic means q^+ and q^-, respectively. If $q^+ \leqq q^-$, then $\Lambda(B) = \Pi(B) = \{c : q^+ \leqq |c| \leqq q^-\}$. If $q^- < q^+$, then $\Lambda(B) = \{c : q^- \leqq |c| \leqq q^+\}$ and*

$$\Pi(B) = \{c : |c| = q^- \text{ or } |c| = q^+\}.$$

Proof. By (3), $i(B)^+ = r(B)^+ = q^+$ and $i(B)^- = r(B)^- = q^-$. All assertions now follow from Theorems 3 and 4.

COROLLARY. *Suppose $a = \lim |s_{-n}|$ and $b = \lim |s_n|$ as $n \to +\infty$. If $b \leqq a$ then $\Lambda(B) = \Pi(B) = \{c : b \leqq |c| \leqq a\}$. If $a < b$ then $\Lambda(B) = \{c : a \leqq |c| \leqq b\}$ and*

$$\Pi(B) = \{c : |c| = a \text{ or } |c| = b\}.$$

EXAMPLE (KELLEY). $s_n = 1$ for negative n, and 2 for positive n. Then

$$\Lambda(B) = \{c : 1 \leqq |c| \leqq 2\} \quad \text{and} \quad \Pi(B) = \{c : |c| = 1 \text{ or } |c| = 2\}.$$

THEOREM 6. $\Pi(L) = \Lambda(L) = \Lambda(R)$.

Proof. The second inequality holds because L and R are adjoint to each other. $\Gamma(L)$, being equal to $\Pi_0(R)$, is either empty or $\{0\}$. Since $\Lambda = \Pi \cup \Gamma$, either $\Pi(L) = \Lambda(L)$ or $\Pi(L) = \Lambda(L) - \{0\}$. But the latter case is impossible. For by the corollary (Kelley) to Theorem 1, either $\Lambda(L) = \{0\}$, in which case $\Pi(L)$ would be empty, or $\Lambda(L)$ is a disk of positive radius, in which case $\Pi(L)$ would fail to be closed.

LEMMA 7. *If i, c, r are any three numbers with $0 \leqq i \leqq c \leqq r$, then there is a positive sequence (s_n) such that*

$$\lim_n \inf_k (s_{k+1} \cdots s_{k+n})^{1/n} = i, \qquad \lim_n \sup_k (s_{k+1} \cdots s_{k+n})^{1/n} = r,$$

and

$$\liminf_n (s_1 \cdots s_n)^{1/n} = c.$$

If i, p, c are any three numbers with $0 \leqq i \leqq p \leqq r$, then there is a positive sequence (s_n) which satisfies the above equalities for i and r, and such that

$$\limsup_n (s_1 \cdots s_n)^{1/n} = p.$$

Proof. We construct the sequences and omit the verifications, which consist of routine analysis; details are in [4].

If $i = r$, let $s_n = r$ for all n (or positive $s_n \to 0$ if $r = 0$).

Suppose $i < r$. Choose a monotone nonincreasing sequence of positive numbers i_k converging to i. (If $i > 0$ we may take $i_k \equiv i$.) Choose a rational sequence $(r_k = p_k/q_k)$, p_k, q_k integers, such that p_k and $q_k - p_k$ tend to infinity and $i_k(r/i_k)^{r_k}$ converges to c. Let (s_n) consist of a sequence of cycles C_k, where each C_k is a sequence of r's of length p_k, followed by a sequence of i_k's of length $q_k - p_k$.

This gives the first required sequence, for i, c, r. To obtain the second, for i, p, r, proceed as before but let the i_k's precede the r's in each cycle C_k.

NOTE. $\Gamma(R)$ is a disk; let $c(R)$ denote its radius. Define $c_1(B)$, $c_2(B)$, $p_1(B)$, and $p_2(B)$ as in the proof of Theorem 4.

THEOREM 8. *If i, c, r are any three numbers with $0 \leqq i \leqq c \leqq r$, then there is an injective right shift R with $i(R) = i$, $c(R) = c$, and $r(R) = r$. If i, p_1, p_2, r are any four numbers with $0 \leqq i \leqq p_1 \leqq p_2 \leqq r$, then there is an injective two-sided shift B with $i(B) = i$, $p_1(B) = p_1$, $p_2(B) = p_2$, and $r(B) = r$. If i^-, c_1, r^-, i^+, c_2, r^+ are any six numbers with $0 \leqq i^- \leqq c_1 \leqq r^- < i^+ \leqq c_2 \leqq r^+$, then there is an injective two-sided shift B with $i^-(B) = i^-$, $c_1(B) = c_1$, $r^-(B) = r^-$, $i^+(B) = i^+$, $c_2(B) = c_2$, and $r^+(B) = r^+$.*

Proof. We exhibit the three asserted shifts by constructing the sequences of weights (s_n), using the two constructions of Lemma 7 in suitable combinations; the verifications are then routine.

(1) Use the first construction (of Lemma 7) directly.

(2) Let $s_0 = 1$. For $n > 0$, use the second construction (Lemma 7), with $i = i$, $p = p_1$, and $r = r$. Let $s_{-n} = t_n$ where (t_n) satisfies the first set of conditions (Lemma 7), with $i = i$, $c = p_2$, and $r = r$.

(3) Let $s_0 = 1$. For $n > 0$, use the first construction of Lemma 7, with $i = i^+$, $c = c_2$, and $r = r^+$. Let $s_{-n} = t_n$ where (t_n) satisfies the second set of conditions (Lemma 7), with $i = i^-$, $p = c_1$, and $r = r^-$.

NOTE. In (2), modification, or caution, may be required if $p_1 = p_2$. For $\Pi_0(B)$ is actually the annulus of convergence of a power series involving the s_n; here it is either a circle of radius $p_1=p_2$, or empty. To ensure that it is indeed the circle, we need only proceed with caution in constructing the sequences (s_n) and (t_n).

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